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Effect of finite conductivity on the inviscid stability of an interface submitted to a high-frequency magnetic field

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The stability of an interface between a liquid metal and an insulating atmosphere, in which an inductor generates a uniform alternating magnetic field, is investigated, with particular attention given to the influence of the electrical conductivity of the liquid. In a range of high frequencies, a quasi-steady approximation is justified, in which the pulsation of the electromagnetic forces is negligible compared with their mean part, and the unsteadiness of the magnetic field only appears through the skin effect. By means of a linear analysis, the influence of the alternating magnetic field on a perturbation of the interface is found to be neutral for wavevectors perpendicular to the magnetic field, and stabilizing for any other orientation. The stabilizing effect is largest when the angle between the wavevector and the magnetic field is zero, and it increases with increasing wavenumber. This effect, maximum for an infinitely conducting medium, quickly decreases with the electrical conductivity.

1. Introduction

To be able to melt, purify, alloy or shape a metal without having resort to any wall, crucible or mould is an old dream of metallurgists. Some recent processes using alternating magnetic fields suggest that this dream is not completely fanciful and might become actual before long; Okress *et al.* (1952) and Sagardia & Segsworth (1977) achieve the levitation melting of important loads of metal; Getselev (1966) control electromagnetically the ingot shapes in the continuous casting of aluminium devices; Etay & Garnier (1983) use high-frequency fields to shape liquid-metal columns. In all these patented techniques, the undesirable and sometimes prohibitive contact between liquid metal and walls is suppressed and chemical or physical contamination of the melt from the crucible is avoided. The equilibrium between the applied electromagnetic forces, gravity, surface tension and fictitious inertial forces determines the position of the liquid-metal–atmosphere interface. The stability of this interface, whose shape will give by cooling and solidifying the desired shape of the final ingot, is a basic requirement in order to achieve successful industrial electromagnetic devices. Through the new mechanisms that it involves, the unsteadiness and the inhomogeneity of the applied magnetic field bring an original character to this stability problem.

The main results concerning the effect of a steady uniform magnetic field on the stability of an interface (Chandrasekhar 1961, p. 428) are independent of the problem we are interested in. They concern the generally stabilizing influence of such fields because of Joule dissipation. In our case, the geometrical deformation of the interface between an insulating atmosphere and an electrically conducting liquid leads, by means of a mutual-influence mechanism, to a perturbation of the induced currents and of the electromagnetic forces in the thin layer near the interface where they are applied. Such an interaction between the position of the interface and an external inductor is specific to alternating magnetic fields and does not appear with steady fields. Likewise, the results established by Roberts (1973) about the effect of a steady magnetic field on Kelvin–Helmholtz instability concern unsteady flows and are independent of the subject of this paper.

General equations for the stability of unsteady parallel flows of a viscous electrically conducting fluid in an unsteady magnetic field were first derived by Drazin (1967). The Joule and viscous dissipation, the mechanisms connected with the pulsation of the magnetic field and with the unsteadiness of the flow, which coexist and interact, are taken into account in this problem, which then becomes highly complex to analyse and mathematically intractable. To get more than meagre results by means of general considerations, such as the power equation which shows the destabilizing effect of the pulsation, Drazin is led to make restrictive approximations and only considers the particular case of a non-dissipative vortex sheet in an alternating magnetic field. In a range of moderate frequencies where the magnetic field may be considered as uniform over a wavelength, the influence of the field is found to be destabilizing. However, neglecting all dissipative mechanisms prevents the drawing of general conclusions, since dissipation may considerably modify the parametric stability.

In order to clarify the effect of the unsteadiness of the magnetic field, our study considers the range of high frequencies ω such that

$$(a)\quad R_\delta = \left(\frac{\mu\sigma}{\omega}\right)^{\frac{1}{2}} V \ll 1,$$

$$(b)\quad N_\delta = \left(\frac{\sigma}{\mu\omega}\right)^{\frac{1}{2}} \frac{B^2}{\rho V} \ll 1,$$

where R_δ and N_δ are respectively the magnetic Reynolds number and the interaction parameter based on the skin depth $\delta = (\frac{1}{2}\mu\sigma\omega)^{-\frac{1}{2}}$. Here μ denotes the magnetic permeability of the fluid, σ its electrical conductivity and ρ its density. B denotes the amplitude of the applied magnetic field and V a typical velocity.

These two conditions define a particular class of MHD problems in which the magnetic field is substantially independent of the motion of the fluid, (a), which is driven by the mean part of the given electromagnetic forces, the unsteady part of which it cannot feel, (b). Condition (a) is classical and implies that the currents induced by the motion of an electrically conducting fluid particle across the magnetic field lines are very small compared with the currents induced by the unsteadiness of the magnetic field, responsible for the skin effect. Condition (b) justifies an important quasi-steady approximation similar to that introduced by Alemany & Moreau (1977) in another context. If ω is high enough it is easy to show that the inertia of fluid prevents it from following electromagnetic oscillations pulsating with frequency 2ω. Indeed the variation ΔV of velocity of a fluid particle submitted during Δt to such quickly pulsating forces is given by

$$\rho\frac{\Delta V}{\Delta t} \sim |\mathbf{J}\times\mathbf{B}| \sim \frac{B^2}{\mu\delta}.$$

During a half-cycle of given sign ($\Delta t \sim \omega^{-1}$), the relative velocity increment is

$$\frac{\Delta V}{V} \sim \frac{B^2}{\rho\mu\delta V\omega} \sim \frac{\sigma B^2\delta}{\rho V} = N_\delta.$$

Thus N_δ characterizes the degree of sensitivity of the fluid to pulsating forces. If $N_\delta \ll 1$, i.e. if ω is high enough, the liquid metal is insensitive to the unsteady part of the electromagnetic forces, and the pulsation of the magnetic field only appears through the skin effect. This explains why high-frequency magnetic fields can lead to steady motions and stationary free surfaces as observed in the studies of Getselev & Martynov (1975), Szekely & Yadoya (1973), and Tarapore & Evans (1976).

In such contexts it is very tempting to consider infinitely high frequencies which prevent any penetration of the magnetic field into the liquid metal, which may then be assumed to be perfectly conducting and submitted to a steady magnetic field with intensity equal to the r.m.s. value of the corresponding unsteady field. Sagardia (1974) studied the stability problem we are interested in, by using this procedure. He found a stabilizing effect of the magnetic field upon disturbances whose wavevector is parallel to the magnetic field, and a neutral effect upon disturbances whose wavevector is perpendicular to the magnetic field: this effect is closely analogous to the stabilizing effect of a d.c. field at the boundary of a perfectly conducting fluid which is well known from studies of instabilities of plasma confinement (see Bateman 1978, p. 54).

However, is such a procedure realistic for the high-frequency fields used in metallurgy, where the skin depth does not reduce to a current sheet? Does the conclusion hold for disturbances whose wavelength is shorter than the skin depth? What is the influence of the perturbation of the electromagnetic forces which is not reducible to the magnetic pressure considered by Sagardia (see Moreau 1980*a*)? It is the aim of this paper to answer these questions by examining this stability problem in the domain of frequencies 10^3–10^5 Hz often encountered in industrial devices. In this domain the order of magnitude of R_δ and N_δ is often less than 10^{-2}, and conditions (*a*) and (*b*) prevail rather than those considered by Drazin. Finite skin depth has been taken into account by Volkov (1962) in the particular case where rapidly travelling magnetic fields are used to contain a heavy conducting fluid against the force of gravity. Stability conditions were derived as function of the spatial period of the travelling wave and of the height of the fluid layer.

The initial equilibrium state and the stability problem are formulated exactly in §2. In §3 the equations of motion linearized with respect to the small amplitude of the disturbance lead to a dispersion relation similar to the dispersion relation obtained in the classical Rayleigh–Taylor or Kelvin–Helmholtz problems, but with a new term in the expression for the external forces which characterizes the influence of the alternating magnetic field on the perturbation of the interface. This new term keeps a constant sign, whatever the values of the parameters, which leads to a stabilizing (or neutral) effect of the magnetic field.

2. Formulation of the problem

Let two inviscid fluids of densities ρ_j $(j = 1, 2)$, be separated by a horizontal boundary at $z = 0$. The subscripts 1 and 2 distinguish respectively the lower and the upper fluid. The two fluids may be at rest or flowing in the same direction Om with uniform velocities $\mathbf{U}_j$. An alternating magnetic field $\mathbf{B}_2 = (B_0 \cos \omega t, 0, 0)$ directed along Ox (figure 1) is applied in the insulating fluid 2. According to the condition

$$R_\delta = \mu\sigma V\delta \sim \left(\frac{\mu\sigma}{\omega}\right)^{\frac{1}{2}} V \ll 1 \tag{1}$$

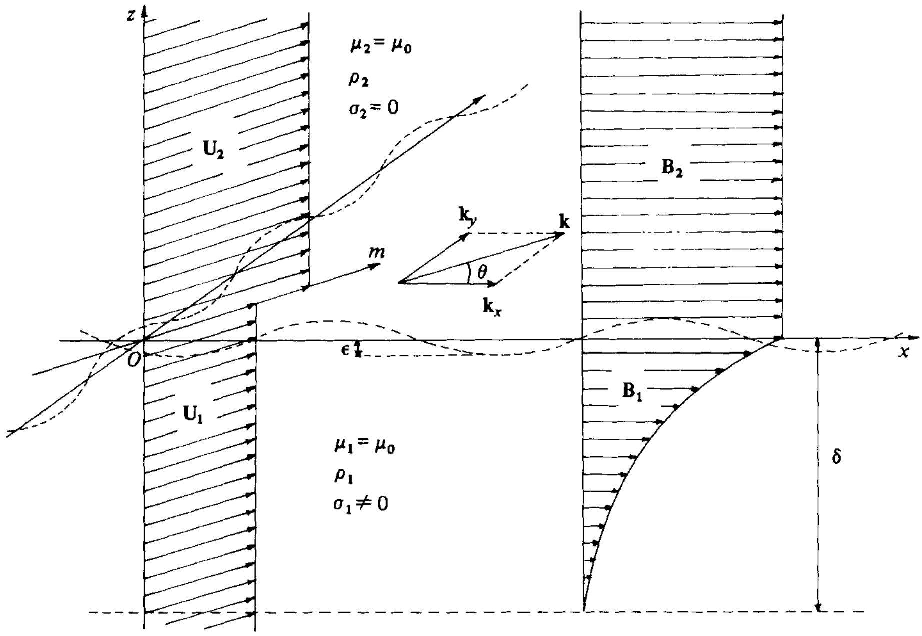

FIGURE 1. Configuration of the problem.

the resulting magnetic field in fluid 1, of finite electrical conductivity σ, governed by a pure diffusion mechanism, satisfies

$$\frac{\partial \mathbf{B}_1}{\partial t} = \frac{1}{\mu\sigma}\nabla^2 \mathbf{B}_1, \tag{2}$$

and is given, in the basic state, by

$$\mathbf{B} = [B_0\, e^{z/\delta} \cos(\omega t + z/\delta), 0, 0]. \tag{3}$$

Let us introduce a perturbation, which may be expanded in terms of normal modes. In the perturbed state the equation of the interface becomes

$$z_0 = \epsilon(t)\, e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{4}$$

where the real vector $\mathbf{k} = (k_x, k_y, 0)$, $\epsilon(t) = \epsilon_0\, e^{ist}$ (with s complex), and it is understood that z_0 is the real part of the complex right-hand side. Because of the quasi-steady approximation, at any time the magnetic field is independent of the motion of the fluid, and is only governed by the interface geometry. Then, to determine the magnetic field it is convenient to consider ϵ as a constant.

The skin effect, through the mutual-influence phenomena it introduces, plays a fundamental part in the stability problem. Hence, to illuminate the phenomena, the skin depth δ, however small it may be, must be taken as the typical lengthscale of the problem. Thus the amplitude ϵ of the perturbation is assumed to be small compared with δ to enable the linearization of the equations with respect to the small parameter ϵ/δ.

The magnetic field, the velocity and the pressure are affected by the perturbation of the interface and may be written as

$$\mathscr{B}_j = \mathbf{B}_j + \mathbf{b}_j, \quad \mathbf{V}_j = \mathbf{U}_j + \mathbf{u}_j, \quad \mathscr{P}_j = P_j + p_j, \tag{5}$$

where the lower-case letters denote the perturbations of the different variables. The boundary conditions, and especially continuity across the interface of the pressure and the normal component of velocity, allow us to express these variables in the form

$$g(z)\, e^{i(st+\mathbf{k}\,.\,\mathbf{r})}. \tag{6}$$

Dealing with the magnetic field, the assumption that the permeability μ of the two fluids is the same imposes the following condition across the perturbed interface between the two fluids of finite electrical conductivity:

$$(\mathscr{B}_1)_{z=z_0} = (\mathscr{B}_2)_{z=z_0}. \tag{7}$$

Then in the perturbed state a discontinuity appears in the x-component of the basic magnetic field, which must be compensated for by the corresponding components of the magnetic-field disturbance:

$$(b_{1x}-b_{2x})_{z=z_0} = (B_{2x}-B_{1x})_{z=z_0}, \tag{8a}$$

$$(b_{1y}-b_{2y})_{z=z_0} = 0, \tag{8b}$$

$$(b_{1z}-b_{2z})_{z=z_0} = 0. \tag{8c}$$

Since the disturbance ϵ is assumed to be small compared with δ, it is possible to linearize the condition on the x-components with respect to ϵ/δ:

$$(B_{2x}-B_{1x})_{z=z_0} = \tfrac{1}{2}B_0(e^{i\omega t}+e^{-i\omega t})-\tfrac{1}{2}B_0\left\{e^{i\omega t}\left[1+\frac{z_0}{\delta}(1+i)\right]+e^{-i\omega t}\left[1+\frac{z_0}{\delta}(1-i)\right]\right\}, \tag{9a}$$

$$(b_{1x}-b_{2x})_{z_0} = -i\,\frac{B_0\epsilon_0}{\sqrt{2}\,\delta}(e^{i\theta_1}-e^{i\theta_2}), \tag{9b}$$

with

$$\theta_1 = (s+\omega)t+\mathbf{k}\,.\,\mathbf{r}-\tfrac{1}{4}\pi, \tag{10a}$$

$$\theta_2 = (s-w)t+\mathbf{k}\,.\,\mathbf{r}+\tfrac{1}{4}\pi. \tag{10b}$$

Because of (9) it is natural to seek the magnetic-field disturbance in the form

$$\mathbf{b}_j = \mathbf{f}_j(z)\, e^{i\theta_1}+\mathbf{g}_j(z)\, e^{i\theta_2}. \tag{11}$$

The Laplace equation for the upper fluid and the diffusion equation for the lower one, together with the condition that all disturbances must vanish far from the interface, lead to the following expressions for the perturbed solenoidal magnetic field:

$$\mathbf{b}_2 = \mathbf{A}\, e^{i\theta_1-kz}+\mathbf{C}\, e^{i\theta_2-\kappa z}, \tag{12a}$$

$$\mathbf{b}_1 = \mathbf{M}\, e^{e\theta_1+\gamma z}+\mathbf{N}\, e^{i\theta_2+\gamma^* z}, \tag{12b}$$

where $\mathbf{A}$, $\mathbf{C}$, $\mathbf{M}$, $\mathbf{N}$ are constant vectors,

$$\gamma^2 = k^2+\frac{2i}{\delta^2} \tag{13}$$

and γ^* denotes the complex conjugate of γ.

Finally the components of the magnetic-field disturbance in the insulating and in the conducting fluid read

$$b_{2x} = i\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}\frac{k_x^2}{k}\left[\frac{1}{\gamma+k}e^{i\theta_1}-\frac{1}{\gamma^*+k}e^{i\theta_2}\right]e^{-kz}, \tag{14a}$$

$$b_{2y} = i\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}\frac{k_x\,k_y}{k}\left[\frac{1}{\gamma+k}e^{i\theta_1}-\frac{1}{\gamma^*+k}e^{i\theta_2}\right]e^{-kz}, \tag{14b}$$

$$b_{2z} = -\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}k_x\left[\frac{1}{\gamma+k}e^{i\theta_1}-\frac{1}{\gamma^*+k}e^{i\theta_2}\right]e^{-kz}, \tag{14c}$$

$$b_{1x} = -i\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}\left[\frac{k_y^2+\gamma k}{k(\gamma+k)}e^{i\theta_1+\gamma z}-\frac{k_y^2+\gamma^* k}{k(\gamma^*+k)}e^{i\theta_2+\gamma^* z}\right], \tag{15a}$$

$$b_{1y} = i\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}\left[\frac{k_x\,k_y}{k(\gamma+k)}e^{i\theta_1+\gamma z}-\frac{k_x\,k_y}{k(\gamma^*+k)}e^{i\theta_2+\gamma^* z}\right], \tag{15b}$$

$$b_{1z} = -\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}\left[\frac{k_x}{\gamma+k}e^{i\theta_1+\gamma z}-\frac{k_x}{\gamma^*+k}e^{i\theta_2+\gamma^* z}\right]. \tag{15c}$$

Interesting information can be deduced from the expressions (14) giving the perturbed magnetic field in the insulating fluid. Let us consider a disturbance whose wavevector is parallel to the magnetic field ($k_y = 0$; $k_x = k$). In this particular case, the magnetic field $\mathscr{B}_2$ is two-dimensional and may be derived from a stream function ψ given by

$$\psi = -\tfrac{1}{2}B_0\,z\,(e^{i\omega t}+e^{-i\omega t})+i\frac{B_0\,\epsilon_0}{\sqrt{2}\,\delta}\left(\frac{1}{\gamma+k}e^{i\theta_1}-\frac{1}{\gamma^*+k}e^{i\theta_2}\right)e^{-kz}. \tag{16}$$

In this asymptotic case of very-short-wavelength disturbances ($k\delta \rightarrow \infty$), this function reduces, to the lowest order in $(k\delta)^{-1}$, to

$$\psi = -\tfrac{1}{2}B_0\,z(e^{i\omega t}+e^{-i\omega t})+i\frac{B_0\,\epsilon_0}{2^{\frac{3}{2}}k\delta}(e^{i\theta_1}-e^{i\theta_2})\,e^{-kz}. \tag{17}$$

When $(k\delta)^{-1} \rightarrow 0$ the stream function coincides with the stream function in the basic state: the magnetic-field lines remain parallel to Ox and cross the peaks of the perturbed interface, which are then submitted to flux variations inducing a restoring force whose effect is to flatten the free surface. A stabilizing effect then arises in this case.

In the asymptotic case of very-long-wavelength disturbances ($k\delta \rightarrow 0$) the stream function reduces to

$$\psi = -\tfrac{1}{2}B_0\,z(e^{i\omega t}+e^{-i\omega t})+\tfrac{1}{2}B_0\,\epsilon_0(e^{i(s+\omega)t}+e^{i(s-\omega)t})\,e^{ikx}\,e^{-kz}, \tag{18}$$

which, using (4), may be written equivalently as

$$\psi = -\tfrac{1}{2}B_0\,z(e^{i\omega t}+e^{-i\omega t})+\tfrac{1}{2}B_0\,z_0(e^{i\omega t}+e^{-i\omega t})\,e^{-kz}. \tag{19}$$

Since $k\epsilon_0 \ll 1$, along the interface $z = z_0$

$$\psi = 0+O(k\epsilon_0)^2. \tag{20}$$

The perturbed interface remains a magnetic-field surface from which the magnetic field diffuses exactly as it did in the basic state, and the induced electromagnetic forces are only modified at the second order in ϵ_0. A neutral effect of the magnetic field is therefore to be expected. The case where the wavevector is perpendicular to the applied magnetic field leads obviously to the same conclusion.

It is to be noticed that these results, closely analogous to those relating to Sargardia's analysis, do not depend on the mechanical properties of the two media: the stability problem is only governed by the interface geometry. The specific influence of the magnetic field ignores the mechanical properties of the two superposed media, and the above conclusions hold for various media, elastic solids or non-Newtonian fluids for example. This fact is confirmed by the detailed analysis of §3.

Ampère's law easily gives the current density $\mathbf{j}$ associated with $\mathbf{b}$. This induced current $\mathbf{j} = \sigma\mathbf{e}$ originates only from the deformation of the interface which introduce the mutual-influence phenomena between the conducting fluid and the external inductor. Other currents induced by the perturbed flow across the basic magnetic field lines and by the basic flow across those of the magnetic-field disturbance are small because of the assumption that the magnetic Reynolds number is small. It is possible to make this assertion precise by distinguishing three complementary parts of the actual perturbation of the current density:

$\mathbf{j}_1 = \sigma\mathbf{e}_1$ induced by the deformation of the interface and independent of any fluid velocity;

$\mathbf{j}_2 = \sigma(\mathbf{e}_2 + \mathbf{u}\times\mathbf{B})$ induced by the perturbed flow in the presence of the basic magnetic field;

$\mathbf{j}_3 = \sigma(\mathbf{e}_3 + \mathbf{U}\times\mathbf{b})$ induced by the basic flow in the presence of the magnetic field disturbance.

Among the two possible lengthscales δ and $\lambda \sim 1/k$, it is always the smaller (which we will denote l) that governs the spatial variation of the magnetic field disturbance. Thus

$$j_1 \sim \frac{B_0\epsilon}{\mu\delta l}, \quad j_2 \sim \sigma B_0 u, \quad j_3 \sim \sigma B_0 U\frac{\epsilon}{\delta}. \tag{21}$$

The vertical component of the velocity perturbation (and, through the continuity equation, each component of the velocity perturbation) is related to the displacement of the interface:

$$w \sim \frac{U\epsilon}{\lambda}, \quad u \sim v \sim \frac{U\epsilon}{l}, \tag{22}$$

which leads to

$$\frac{j_2}{j_1} \sim R_\delta, \quad \frac{j_3}{j_1} \sim R_\delta\frac{l}{\delta}. \tag{23}$$

Therefore j_2 and j_3 are negligible compared with j_1.

The linearized expressions for the perturbation of the electromagnetic forces (in which the suffix 1 is suppressed because only the conducting fluid is submitted to such forces) read

$$\mathbf{f} = (Jb_z, j_z B_x, -Jb_x - j_y B_x), \tag{24}$$

where

$$J = -\frac{B_0\sqrt{2}}{\mu\delta}e^{z/\delta}\sin\left(\frac{z}{\delta} + \omega t - \tfrac{1}{4}\pi\right)$$

follows from (3) as the unperturbed eddy currents, and where its disturbance is written as

$$j_y = i\frac{B_0\epsilon_0}{\sqrt{2}\,\mu\delta}\left(\frac{k_x^2 - \gamma k}{k}e^{i\theta_1 + \gamma z} - \frac{k_x^2 - \gamma^* k}{k}e^{i\theta_2 + \gamma^* z}\right), \tag{25a}$$

$$j_z = -\frac{B_0\epsilon_0}{\sqrt{2}\,\mu\delta}k_y(e^{i\theta_1 + \gamma z} - e^{i\theta_2 + \gamma^* z}). \tag{25b}$$

The condition $N_\delta \ll 1$ implies, as we demonstrated in §1, that the conducting fluid cannot follow oscillating forces with frequency 2ω, and that only the average component (with respect to the timescale ω^{-1}) of electromagnetic forces is to be retained. The forces are therefore given by

$$f_x = i\frac{B_0^2 \epsilon_0 k_x}{2\mu\delta^2}\left(\frac{e^{\alpha z}}{\gamma+k}+\frac{e^{\alpha^* z}}{\gamma^*+k}\right)e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{26a}$$

$$f_y = -\frac{B_0^2 \epsilon_0 k_y}{4\mu\delta}[(1-i)\,e^{\alpha z}-(1+i)\,e^{\alpha^* z}]\,e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{26b}$$

$$f_z = \frac{B_0^2 \epsilon_0}{2\mu\delta^2}\left\{\left[\frac{k_y^2+\gamma k}{k(\gamma+k)}-\frac{1+i}{2}\frac{\delta}{k}(k_x^2-\gamma k)\right]e^{\alpha z}+\left[\frac{k_y^2+\gamma^* k}{k(\gamma+k)}-\frac{1-i}{2}\frac{\delta}{k}(k_x^2-\gamma k)\right]e^{\alpha^* z}\right\}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{26c}$$

where $\alpha = \gamma+(1-i)/\delta$.

3. Dispersion relation and analytical results

The equations of motion of the two incompressible and inviscid fluids subjected to electromagnetic forces are

$$\rho_j\frac{\partial \mathbf{V}_j}{\partial t}+\rho_j(\mathbf{V}_j\nabla)\,\mathbf{V}_j = -\nabla P_j+\mathbf{F}_j. \tag{27a}$$

$$\nabla.\mathbf{V}_j = 0. \tag{27b}$$

After introduction of perturbations (defined in (5)) proportional to $\epsilon = \epsilon_0 e^{i(st+\mathbf{k}\cdot\mathbf{r})}$ and linearization with respect to ϵ_0/δ, elimination of the pressure leads to a second-order differential equation for the amplitude of the vertical component of the disturbance velocity $w_j(z)$:

$$\rho_j(s+\mathbf{k}.\mathbf{U}_j)\,(D^2-k^2)\,w_j = ik^2 f_{jz}-D(\mathbf{k}.\mathbf{f}_j), \tag{28}$$

with $D \equiv \partial/\partial z$.

The boundary conditions on w_j demand that it vanish far from the interface in the two fluids:

$$w_1(z\rightarrow-\infty)\rightarrow 0, \quad w_2(z\rightarrow+\infty)\rightarrow 0, \tag{29}$$

and that the displacement of any point of the interface be unique and compatible with the expression (4) for $z_0(x,y,t)$ (see Chandrasekhar 1961):

$$\frac{w_1(z_0)}{s+\mathbf{k}.\mathbf{U}_1} = \frac{w_2(z_0)}{s+\mathbf{k}.\mathbf{U}_2} = i\epsilon_0\, e^{i(st+\mathbf{k}\cdot\mathbf{r})}. \tag{30}$$

Then the dispersion relation follows from the condition of continuity of normal stress, which can be written:

$$(P_2+p_2)_{z_0} = (P_1+p_1)_{z_0}-Tk^2 z_0, \tag{31}$$

where T is the surface tension.

After some algebra the expressions for $w_1(z)$ and $w_2(z)$, which satisfy (22)–(24), are to the first order in ϵ

$$w_1(z) = \epsilon\{e^{kz}\,i(s+\mathbf{k}.\mathbf{U}_1)-w_0(0)+w_0(z)\}, \tag{32a}$$

$$w_2(z) = i\epsilon(s+\mathbf{k}.\mathbf{U}_2)\,e^{-kz}, \tag{32b}$$

where $w_0(z)$ represents the contribution of the particular solution of (28) with the expressions (26) for the electromagnetic forces:

$$w_0(z) = -\frac{iB_0^2 k_x^2}{2\mu\delta^2\rho_1(s+\mathbf{k}.\mathbf{U}_1)}\left[\frac{e^{\alpha z}}{\gamma(\gamma+k)}+\frac{e^{\alpha^* z}}{\gamma^*(\gamma^*+k)}\right]. \tag{33}$$

Taking the divergence of the equations of motion leads to the equation for the pressure perturbation:

$$ik^2 p_j = \rho_j(s+\mathbf{k}\,.\,\mathbf{U}_j)Dw_j + \mathbf{k}\,.\,\mathbf{f}_j. \tag{34}$$

In the unperturbed state the pressure satisfies

$$P_2+\rho_2 gz = P_1+\rho_1 gz + \left\langle \frac{B_1^2(z)}{2\mu} \right\rangle - \left\langle \frac{B_1^2(0)}{2\mu} \right\rangle. \tag{35}$$

This expression is deduced from the hydrostatic equilibrium of the insulating fluid and from the magnetostatic equilibrium of the conducting one; the angle brackets $\langle\,\rangle$ mean that we only retain the mean value over a period according to our quasi-steady approximation.

Substituting (35) in (31) gives the condition at the interface for the pressure perturbation:

$$(p_2-p_1)_{z_0} - (\rho_2-\rho_1)\, gz_0 + \frac{B_0^2}{2\mu}\frac{z_0}{\delta} + Tk^2 z_0 = 0. \tag{36}$$

Here the linearized expression for the magnetic pressure difference

$$\left\langle \frac{B_1^2(z_0)}{2\mu} \right\rangle - \left\langle \frac{B_1^2(0)}{2\mu} \right\rangle = z_0 \langle B_1(0)\, J_y(0) \rangle = \frac{B_0^2}{2\mu}\frac{z_0}{\delta} \tag{37}$$

has been taken into account, and introduces in (36) the influence of the basic electromagnetic forces on the deformation of the interface. The dispersion relation is deduced from (34) and (36):

$$\rho_1(s+\mathbf{k}\,.\,\mathbf{U}_1)^2 + \rho_2(s+\mathbf{k}\,.\,\mathbf{U}_2)^2 + gk(\rho_2-\rho_1) - Tk^3 - F(k,\theta) = 0. \tag{38}$$

This relation is quite similar to its analogue in classical hydrodynamics, but a new term appears which is the net result of the electromagnetic effects:

$$F(k,0) = \frac{B_0^2 k}{2\mu\delta} + i\frac{\rho_1(s+\mathbf{k}\,.\,\mathbf{U}_1)}{k}(w_0'(0) - kw_0(0)) + \frac{i}{k}(\mathbf{k}\,.\,\mathbf{f}(0)). \tag{39}$$

This function can be expressed in terms of the dimensionless wavenumber $x = k\delta$ and the angle θ between the applied magnetic field and the wavevector:

$$F(k,\theta) = \frac{B_0^2 k^2 \cos^2\theta}{2\mu} F(x), \tag{40}$$

where

$$F(x) = \frac{[(x^4+4)^{\frac{1}{2}}+x^2]^{\frac{1}{2}} + (1-x)\,[(x^4+4)^{\frac{1}{2}}-x^2]^{\frac{1}{2}}}{\sqrt{2}\,(x^4+4)^{\frac{1}{2}}} \quad (F(x) \sim 1/x \quad \text{as} \quad x \to \infty). \tag{41}$$

A parallel may be drawn between the electromagnetic term and the term relating to surface tension: both are independent of the velocity field because they only depend on the geometry of the interface and are therefore independent of the mechanical properties of the fluids.

Sagardia's assumption, which only considers an infinitely conducting fluid, leads to the following corresponding electromagnetic term:

$$\frac{B_0^2 k^2 \cos^2\theta}{2\mu}. \tag{42}$$

This is in perfect agreement with our analysis, since $F(x)$ tends to unity when x tends to zero with δ (see figure 2).

The increment s is determined by a quadratic equation, which has in general two

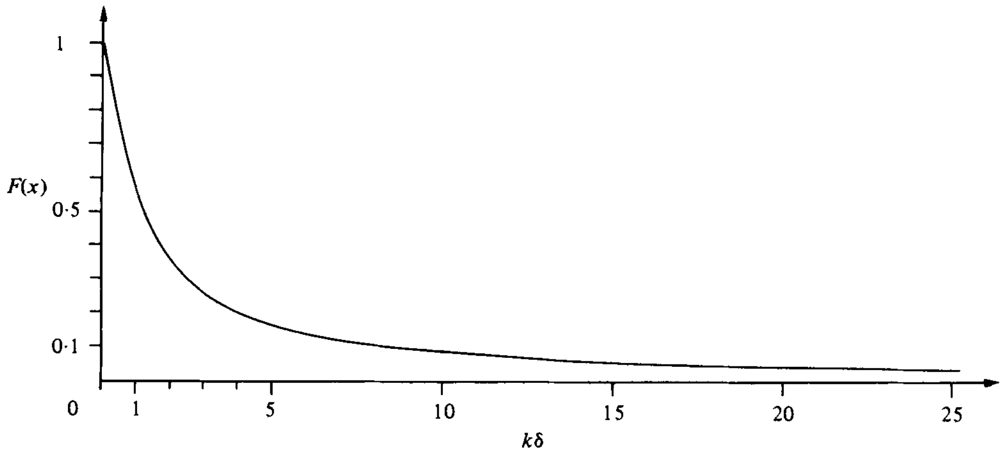

FIGURE 2. Function $F(x)$ plotted against $x = k\delta$.

complex roots: one leads to an amplification, the other to a damping down. Hence to prevent any perturbation form being amplified it is necessary that the discriminant of (38) be positive or zero to ensure that s be a real root:

$$-\frac{\rho_1\rho_2}{\rho_1+\rho_2}k^2(U_1-U_2)^2\cos^2\theta_1+gk(\rho_1-\rho_2)+Tk^3+F(k,\theta)\geqslant 0, \tag{43}$$

where θ_1 denotes the angle $(\mathbf{U}_1, \mathbf{k})$.

The influence of each of the four terms involved in (36) upon the evolution of a disturbance is as follows. Whatever the values of U_1, U_2, ρ_1, ρ_2, θ_1, the first term is always negative and therefore has a destabilizing effect. This term is responsible for the Kelvin–Helmholtz instability. The second term, whose sign is given by the difference $\rho_1-\rho_2$, expresses the stabilizing or destabilizing effect of gravity according to whether ρ_1 is greater or less than ρ_2. This term is responsible for the Rayleigh–Taylor instability. The third term is always positive. Thus the surface tension, which provides a restoring force to any displacement of the interface that tends to bend it, has a stabilizing effect. The stronger the local curvature of the interface, the stronger is the stabilizing effect. Such an effect is specific to the plane geometry, and it becomes destabilizing in an axisymmetric geometry for symmetric varicose deformations with wavelengths exceeding the circumference of the cylinder (Chandrasekhar 1961). The last term characterizes the presence of the electromagnetic field, whose influence appears to be always stabilizing ($F(x) \geqq 0$ and $F(x) \sim x^{-1}$ when $x \rightarrow \infty$) except in the limit of very large wavelengths ($x \rightarrow 0$ when $k \rightarrow 0$) and of wavevectors perpendicular to the magnetic field ($\theta = \frac{1}{2}\pi$) for which $F(k, \theta)$ is zero. This effect is maximum when the skin depth is zero, and is quickly decreasing when the skin depth increases (figure 2).

It is to be noticed that, if the effect predicted by the analysis supposing a zero skin depth is qualitatively good, from a quantitative point of view important corrections are to be made when realistic cases of finite conductivities are considered. To be more precise let us consider the typical example where the skin depth is 1·6 cm (i.e. frequency of the order of 1 kHz in liquid metals). To stabilize disturbances of wavelength of 1 cm upon which surface tension is ineffective the required power is $F^{-1}(x = 10)$ times greater (i.e. 10 times greater) than that predicted by Sagardia's theory.

In the context of the levitation problem, as studied by Sagardia, and of the magnetic shaping of liquid–metal columns, as studied by Etay & Garnier, the constant-field analysis is only valid for wavelengths smaller than the scale of variation of the applied magnetic field, typically a few centimetres. For wavelengths shorter than a few tenths of a millimetre the stabilizing effect of surface tension prevails over that of the magnetic field. Therefore the range of wavelengths in which the results of this analysis are most relevant is 10^{-2}–10 cm.

4. Conclusion

The effect of a uniform alternating magnetic field on the stability of a plane interface between a conducting liquid and an insulating atmosphere has been investigated with particular attention given to the analysis of the influence of the electrical conductivity of the liquid. This effect is always stabilizing except for large-wavelength disturbances of the interface and for disturbances whose wave-vectors are perpendicular to the applied magnetic field. For a given wavelength this effect is maximum when the skin depth is zero (infinite electrical conductivity), and decreases when the skin depth increases. Differences then appear between the intensities of the alternating magnetic fields able to stabilize a given configuration deduced from theories assuming a zero skin depth and the theory taking skin effect into account. In some cases these differences may be so important that the order of magnitude of the power necessary to generate the magnetic field, whose intensity is deduced from the asymptotic theory, is wrong.

Though the analysis is limited to plane interfaces, the results are valuable in any cylindrical geometries if the skin depth δ remains small compared with the typical radius R of the conducting medium. This study can be extended to the general case of liquid-metal jets for any values of the parameter δ/R. In a cylindrical geometry surface tension tends to increase the local curvature of the perturbed interface and can compete with the stabilizing effect of the magnetic field.

In the scope of industrial applications it is interesting to estimate the difference between a.c. devices and d.c. devices with regard to their respective stabilizing influences. The effect of an axial magnetic field on the capillary instability of a liquid-metal jet gives a good example. Chandrasekhar's analysis conclusion for a d.c. magnetic field is that 'for experiments with mercury, magnetic fields of the order of 10^4 G will be needed to demonstrate the stabilizing effect of an axial magnetic field' and 310^5 G, a really enormous field, is necessary 'to overcome significantly the paramount effects of finite resistivity'. With an a.c. axial magnetic field and mercury only 10^3 G are sufficient to obtain a good stabilization. This illustrates the advantage of a.c. fields over d.c. fields. Basically the difference is clear: the typical parameter of d.c. field effects is the magnetic Reynolds number R_{m}, which is always small, whereas a.c. magnetic-field effects are governed by the parameter $R_{\omega} = \mu\sigma\omega L^2$, which may be large since high frequencies can compensate for small conductivities.

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Addendum

While this paper was under revision, another closely related paper was submitted to, and published by, this Journal (McHale & Melcher 1982), in which experimentally observed instabilities of a free surface of liquid metal are reported. A theory is also proposed, which concludes that a high-frequency magnetic field has a destabilizing influence when the non-dimensional number $B_0^2/\mu\rho\nu\omega$ (which is the square of the Hartmann number $M = (\sigma/\rho\nu)^{\frac{1}{2}} B_0 \delta$ based upon the skin depth) is larger than some critical value (of the order of 100 when the boundaries are rigid, and 70 when a free surface is present).

In order to clarify the difference between this theory and that of the present paper, which are concerned with two different problems, a few statements have to be made.

(i) McHale & Melcher do not neglect eddy currents due to the interaction of the velocity and the applied magnetic field, even though these are extremely small (of the order of the magnetic Reynolds number $R_{m\delta} = \mu\sigma v\delta$) when compared with the eddy current due to the pulsation. By contrast, we do neglect them (see §2), since it is our belief that in the present context any phenomenon on the laboratory scale should be explicable within the framework of an asymptotic theory at zero magnetic Reynolds number.

(ii) In the present paper, disturbances of any quantities are just those generated by some deformation of the interface $z_0(\mathbf{r}, t)$. They are proportional to z_0, and they would reduce to zero if the fluid were bounded by a rigid plane. In the McHale & Melcher theory the deformation of the upper free surface, when present, is only introducing some minor changes in the numerical values, but does not affect appreciably the main behaviour of the solution of the dispersion equation.

(iii) McHale & Melcher pointed out discrepancies between the observed (1–10 s) and the predicted (10^2–10^3 s) growth rates, which lead them to believe that the observed motion could be driven by thermal convection due to Joule heating.

Finally it is the feeling of the authors of the present paper, who also conducted numerous experiments with liquid metals in the presence of a.c. magnetic fields, that the instabilities reported by McHale & Melcher do occur in some circumstances. But we consider that the proper theory, the governing mechanisms, and the precise conditions of instability, have still to be discovered. The present paper, demonstrating the stabilizing influence of an a.c. magnetic field on a disturbed free surface, is just one step in this search.

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